

# Quasi-exactly solvable Fokker-Planck equations

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We consider exact and quasi-exact solvability of the one-dimensional Fokker-Planck equation based on the connection between the Fokker-Planck equation and the Schrödinger equation. A unified consideration of these two types of solvability is given from the viewpoint of prepotential together with Bethe ansatz equations. Quasi-exactly solvable Fokker-Planck equations related to the  $sl(2)$ -based systems in Turbiner's classification are listed. We also present one  $sl(2)$ -based example which is not listed in Turbiner's scheme.

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## I. INTRODUCTION

Without doubt the Fokker-Planck (FP) equation is one of the basic tools used to deal with fluctuations in various kinds of systems [1]. It has found applications in such diverse areas as physics, astrophysics, chemistry, biology, finance, etc. Owing to its wide applicability, various methods of finding exact and approximate solutions of the FP equations have been developed.

One of the methods of solving the FP equation is to transform the FP equation into a Schrödinger-like equation, and then solve the eigenvalue problem of the latter. This method, to be called the method of eigenfunction expansion, is useful when the associated Schrödinger equation is exactly solvable; for example with infinite square well, harmonic oscillator potentials, etc. Several FP equations have been exactly solved in this way [1].

Unfortunately exactly solvable Schrödinger equations are rather limited in number, and hence the method of eigenfunction expansion is quite restricted. However, in real calculations more often than not a knowledge of a large but finite number of eigenfunctions is sufficient for a good approximate result. This being so, it would render the method of eigenfunction expansion more useful. Recently, in non-relativistic quantum mechanics a new class of potentials which are intermediate to exactly solvable ones and non-solvable ones has been found. These are called quasi-exactly solvable (QES) problems for which it is possible to determine algebraically a part of the spectrum (eigenvalues and eigenfunctions) but not the whole spectrum [2, 3, 4, 5, 6]. These QES Schrödinger equations thus extend the range of applicability of the eigenfunction expansion.

In this paper, we shall classify  $sl(2)$ -based QES FP equations according to their associated QES Schrödinger equations listed in [3], and present a case which is not listed in [3]. We first discuss the connection between the FP and Schrödinger equations in Section II. In Section III the basics of the exact and quasi-exact solvability of Schrödinger equations are reviewed from the viewpoint of prepotential together with Bethe ansatz equations, and the way to construct the corresponding QES FP equations is presented. Some examples of QES FP equations constructed in this manner are discussed in Section IV. Section V concludes the paper.

## II. FOKKER-PLANCK AND SCHRÖDINGER EQUATIONS

In one dimension, the FP equation of the probability density  $P(x, t)$  is [1]

$$\begin{aligned}\frac{\partial}{\partial t}P(x, t) &= \mathcal{L}P(x, t), \\ \mathcal{L} &\equiv -\frac{\partial}{\partial x}D^{(1)}(x) + \frac{\partial^2}{\partial x^2}D^{(2)}(x).\end{aligned}\tag{1}$$

The functions  $D^{(1)}(x)$  and  $D^{(2)}(x)$  in the FP operator  $\mathcal{L}$  are, respectively, the drift and the diffusion coefficient (we consider only time-independent case). The drift coefficient represents the external force acting on the particle, while the diffusion coefficient accounts for the effect of fluctuation. The drift coefficient is usually expressed in terms of a drift potential  $\Phi(x)$  according to  $D^{(1)}(x) = -\Phi'(x)$ , where the prime denotes the derivative with respect to  $x$ . Without loss of generality, in what follows we shall take  $D^{(2)} = 1$ . The stationary solution of the FP equation is  $P_0(x) = \exp(-\Phi(x))$ , with  $\int P_0(x) dx = 1$ .

The FP equation is closely related to the Schrödinger equation [1]. To see this, let us define  $\psi(x, t) \equiv e^{\Phi/2} P(x, t)$ . Substituting this into the FP equation, we find that  $\psi$  satisfies the Schrödinger-like equation:

$$\frac{\partial \psi}{\partial t} = \left( \frac{\partial^2}{\partial x^2} - \left( \frac{\Phi'}{2} \right)^2 + \left( \frac{\Phi}{2} \right)'' \right) \psi. \quad (2)$$

As we shall see in what follows, the function  $\Phi/2$  plays a fundamental role in our discussions. So let it be denoted by  $W(x) \equiv \Phi(x)/2$ . Following [7], we shall call  $W(x)$  the prepotential. Letting  $\psi(x, t) = \exp(-\lambda t)\phi(x)$  and  $\phi_0 \equiv \exp(-W)$ , we obtain

$$H\phi = \lambda\phi, \quad (3)$$

where

$$\begin{aligned} H &\equiv -\phi_0^{-1} \mathcal{L} \phi_0 \\ &= -\frac{\partial^2}{\partial x^2} + V(x), \\ V(x) &= W'(x)^2 - W''(x). \end{aligned} \quad (4)$$

Thus  $\phi$  satisfies the time-independent Schrödinger equation with Hamiltonian  $H$  and eigenvalue  $\lambda$ , and  $\phi_0$  is the zero mode of  $H$ :  $H\phi_0 = 0$ . Since only derivatives of  $W(x)$  appear in  $V(x)$ ,  $W(x)$  is defined only up to an additive constant. We choose the constant in such a way as to normalize  $\phi_0(x)$  properly,  $\int \phi_0(x)^2 dx = 1$ . For simplicity of presentation, we consider the cases in which the ground state wave functions are square integrable, that is the corresponding FP operators have the normalizable stationary distribution.

Now comes the essence of the method of eigenfunction expansion for solving FP equations. If all the eigenfunctions  $\phi_n$  ( $n = 0, 1, 2, \dots$ ) of  $H$  with eigenvalues  $\lambda_n$  are solved, then the eigenfunctions  $P_n(x)$  of  $\mathcal{L}$  corresponding to the eigenvalue  $-\lambda_n$  is  $P_n(x) = \phi_0(x)\phi_n(x)$ . The stationary distribution is  $P_0 = \phi_0^2 = \exp(-2W)$ , which is obviously non-negative, and is the zero mode of  $\mathcal{L}$ :  $\mathcal{L}P_0 = 0$ . Any positive definite initial probability density  $P(x, 0)$  can be expanded as  $P(x, 0) = \phi_0(x) \sum_n c_n \phi_n(x)$ , with constant coefficients  $c_n$  ( $n = 0, 1, \dots$ )

$$c_n = \int_{-\infty}^{\infty} \phi_n(x) (\phi_0^{-1}(x) P(x, 0)) dx. \quad (5)$$

Then at any later time  $t$ , the solution of the FP equation is  $P(x, t) = \phi_0(x) \sum_n c_n \phi_n(x) \exp(-\lambda_n t)$ .

It is now evident that the exact and quasi-exact solvability of FP equations are directly related to the exact and quasi-exact solvability of the corresponding Schrödinger equations.

We shall classify the types of exactly solvable and QES FP equations associated with Schrödinger equations which are factorizable (or supersymmetric). For this, it is desirable to have a unified approach to deal with both kinds of solvability. It appears to us that the approach based on the so-called prepotential serves this purpose well. We shall therefore first review exact and quasi-exact solvability from this viewpoint below.

### III. EXACT AND QUASI-EXACT SOLVABILITY FROM THE POINT OF VIEW OF PREPOTENTIAL WITH BETHE ANSATZ EQUATIONS

Let us now briefly review the essence of quasi-exact and exact solvability of Schrödinger equations. We choose to proceed from the point of view of prepotentials together with the Bethe ansatz equations [7] (see also [8]). The merit of this approach is that one needs not to first assume the form of the potential of the system concerned, as all information about the system is contained in the prepotential and the solutions, or roots, of the Bethe ansatz equations [9]. Also, in this approach exact and quasi-exact solvability can be treated on the same footing. Furthermore, such approach facilitates extension of the QES theory from the Schrödinger equation to equations for multi-component wave functions, such as the Pauli and the Dirac equation [8].

Suppose  $\phi_0 \equiv e^{-W_0(x)}$  is the ground state, with zero energy, of a Hamiltonian  $H_0$ :  $H_0\phi_0 = 0$ . This is the simplest example of quasi-exact solvability. This implies that the potential  $V_0$  is completely determined by  $W_0$ :  $V_0 = W_0'^2 - W_0''$ , and consequently, the Hamiltonian is factorizable:

$$H_0 = \left( -\frac{d}{dx} + W_0' \right) \left( \frac{d}{dx} + W_0' \right). \quad (6)$$

This fact can also be considered as the very base of supersymmetric quantum mechanics. Consider now a wave function  $\phi_N$  ( $N$ : positive integer) which is related to  $\phi_0$  of  $H_0$  by  $\phi_N = \phi_0 \tilde{\phi}_N$ :

$$\tilde{\phi}_N = (z - z_1)(z - z_2) \cdots (z - z_N). \quad (7)$$

Here  $z = z(x)$  is some function of  $x$ . The function  $\tilde{\phi}_N$  is a polynomial in an  $(N + 1)$ -dimensional Hilbert space with the basis  $\langle 1, z, z^2, \dots, z^N \rangle$ . One can rewrite  $\phi_N$  as

$$\phi_N = \exp(-W_N(x, \{z_k\})), \quad (8)$$

with

$$W_N(x, \{z_k\}) = W_0(x) - \sum_{k=1}^N \ln |z(x) - z_k|. \quad (9)$$

Now we form  $W_N'^2 - W_N''$ . This results in a Riccati equation

$$W_N'^2(x, \{z_k\}) - W_N''(x, \{z_k\}) = V_N(x, \{z_k\}) - \lambda_N(\{z_k\}), \quad (10)$$

where  $V_N$  is in general a function of  $N$ ,  $x$  and  $\{z_k\}$ , and  $\lambda_N$  is a real constant depending on  $N$  and  $\{z_k\}$  (dependence of  $V_N$  and  $\lambda_N$  on other parameters of  $H_0$  is understood). It should be noted that the definition of  $V_N$  and  $\lambda_N$  has ambiguity, namely, they can both be shifted by an arbitrary real constant  $\alpha$ :

$$V_N \rightarrow V_N + \alpha, \quad \lambda_N \rightarrow \lambda_N + \alpha. \quad (11)$$

This corresponds to the redefinition of the zero point of  $V_N$ . For the theory we are considering, the r.h.s. of (10) is generally a meromorphic function of  $z$  with at most simple poles. Let us demand that the residues of the simple poles,  $z_k$ ,  $k = 1, \dots, N$  should all vanish. This will result in a set of algebraic equations which the parameters  $\{z_k\}$  must satisfy. These equations are called the Bethe ansatz equations for  $\{z_k\}$ . For each value of  $N$  ( $N = 0, 1, 2, \dots$ ), there are  $N + 1$  possible sets of solution  $\{z_k\}$  of the Bethe ansatz equations

With  $\{z_k\}$  satisfying the Bethe ansatz equations, the r.h.s. of (10) will have no simple poles at  $\{z_k\}$  but it still generally depends on  $\{z_k\}$ . Suppose now that the  $N + 1$  sets of roots give the same form of the potential  $V_N(x, \{z_k\}) = V_N(x)$  in the r.h.s. of (10), but with  $N + 1$  values of  $\lambda_N$ . Then the Riccati equation (10) implies that we have a new eigenvalue problem with Hamiltonian  $\mathcal{H}_N$ :

$$\begin{aligned} \mathcal{H}_N &\equiv -\frac{d^2}{dx^2} + V_N, \\ \mathcal{H}_N \phi_N &= \lambda_N \phi_N. \end{aligned} \quad (12)$$

The  $N + 1$  sets of  $\{z_k\}$  define  $N + 1$  eigenfunctions  $\phi_N$  of  $\mathcal{H}_N$  with eigenvalue  $\lambda_N$ . In other words,  $N + 1$  eigenfunctions and eigenvalues of the Hamiltonian  $\mathcal{H}_N$  are solvable algebraically, and  $\mathcal{H}_N$  is said to be QES. If the wave function  $\phi_N^{(0)} \equiv \exp(-W_N^{(0)})$  corresponding to the lowest value  $\lambda_N = \lambda_N^{(0)}$  has no node, then it is the ground state of  $\mathcal{H}_N$ , and is annihilated by the Hamiltonian  $H_N$ :

$$H_N \equiv \mathcal{H}_N - \lambda_N^{(0)} = -\frac{d^2}{dx^2} + W_N^{(0)'} - W_N^{(0)''}, \quad (13)$$

$$H_N \phi_N^{(0)} = 0. \quad (14)$$

Now if  $V_N \equiv V$  is not only independent of the solutions of the Bethe ansatz equations  $\{z_k\}$ , but is also independent of  $N$ , then the Hamiltonian  $H$  is exactly solvable, as solvable states can be found for any integral value of  $N$ .

From the Schrödinger equation  $\mathcal{H}_N \phi_N = \lambda_N \phi_N$ , it is also seen that the polynomial part  $\tilde{\phi}_N$  satisfies the equation  $h_N \tilde{\phi}_N = \lambda_N \tilde{\phi}_N$ , where  $h_N \equiv \phi_0^{-1} \mathcal{H}_N \phi_0$ , and  $\phi_0$  is the ground state of  $H_0$  corresponding to  $N = 0$ . From the previous discussions, we know that  $h_N$  has an algebraic sector with  $N + 1$  eigenvalues and eigenfunctions, which are polynomials in an  $(N + 1)$ -dimensional Hilbert space with the basis  $\langle 1, z, z^2, \dots, z^N \rangle$ . The quasi-exact solvability of  $H_N$  is said to have a Lie-algebraic origin, if  $h_N$  can be expressed as

$$h_N = \sum C_{ab} J^a J^b + \sum C_a J^a + \text{constant}, \quad (15)$$

where  $C_{ab}$ ,  $C_a$  are constant coefficients, and the  $J^a$  are the generators of some Lie algebra. QES Hamiltonians based on the Lie-algebra  $sl(2)$  have been classified in [3, 4].

There exists a different situation when the  $N + 1$  sets of roots  $\{z_k\}$  are plugged into the l.h.s. of (10). Instead of resulting in the r.h.s. of (10) a single potential  $V_N$  with  $N + 1$  values of  $\lambda_N$ ,  $N + 1$  potentials  $V_N$  with a single value of  $\lambda_N$  are obtained, when the zero point of the eigenvalue is properly adjusted. These are QES systems for which a set of  $N + 1$  potentials differing by the values of parameters have the same eigenvalue of the  $j$ -th eigenstate in the  $j$ -th potential ( $j = 1, 2, \dots, N + 1$ ). Such kind of QES systems were termed the second type QES systems, and those discussed previously were called the first type in [3]. For the second type QES quantum systems which have  $sl(2)$  Lie-algebraic origin, it is  $(h_N - E)/\rho(x)$  instead of  $h_N$ , where  $\rho(x)$  is some  $x$ -dependent scale factor, that is expressible in terms of certain quadratic combination of the generators  $J^a$ 's. In this sense, one can say that for the first type systems, the scale factors  $\rho$  are simply some real constants.

Combining the discussions in this section and those in the preceding section, we arrive at a recipe of constructing QES FP equation from a known QES Schrödinger equation. We shall be interested only in the first type QES systems, since for the second type QES systems the corresponding  $W_N$  only define  $N + 1$  FP systems with  $N + 1$  different drift potentials  $\Phi_N = 2W_N$  having the same eigenvalue. While these are QES FP systems, it is not useful for good approximate calculations using the method of eigenfunction expansion as only one eigenstate is known.

Suppose a prepotential  $W_0$  and a corresponding set of Bethe ansatz equations define a first type QES Schrödinger equation in the sense discussed above. Then the drift potential  $\Phi_0 = 2W_0$  defines a QES FP equation with only one exactly solved state  $\phi_0$ , and the stationary solution of the FP equation is  $P_0(x) = \phi_0^2$ . To obtain QES FP equations admitting a larger number of solvable states, we construct a new prepotential  $W_N$  from  $W_0$  and the  $N$  roots  $\{z_1, z_2, \dots, z_N\}$  of the Bethe ansatz equations according to (9). The set of roots  $\{z_k\}$  to be chosen is the set for which the value of eigenvalue  $\lambda_N$  is the lowest. If the corresponding wave function  $\phi_N^{(0)}$  has no node, one can use it to define a corresponding FP operator:

$$\mathcal{L}_N \equiv -\phi_N^{(0)} H_N \phi_N^{(0)-1}. \quad (16)$$

Then  $\mathcal{L}_N$  defines a QES FP equation having  $N + 1$  solvable states, with a drift potential  $\Phi_N = 2W_N^{(0)}$ . As mentioned before, if  $H_N$  is independent of  $N$ , then  $H_N$  is exactly solvable, and so is the corresponding FP equation.

We shall illustrate this procedure by some examples below.

## IV. EXAMPLES

### A. Exactly solvable cases

It is clear that FP equations transformable to exactly solvable Schrödinger equations can be exactly solved. Particularly, all the shape-invariant potentials in supersymmetric quantum mechanics, as listed in Table 4.1 of [10], or equivalently, Table 5.1 in [11], give the corresponding exactly solvable FP systems. One needs only to link the prepotential  $W$  [12] in the Schrödinger system with the drift potential  $\Phi$ , or drift coefficient  $D^{(1)}$ , in the FP system according to the relation  $\Phi = 2W$  and  $D^{(1)} = -\Phi'$ . We shall not list all of them here, but simply mention that the FP equation for the Ornstein-Uhlenbeck process corresponds to the shifted oscillator potential in quantum mechanics, and that for the Rayleigh process corresponds to the three-dimensional oscillator potential.

### B. $sl(2)$ -based QES cases listed in Turbiner's classification

Similarly, one can identify all QES FP equations which are based on the  $sl(2)$  Lie algebra from the list of ten classes depicted in Table 1 of [3]. Of the ten classes, classes II, III, V, VIII and IX belong to the second type QES systems which, as discussed previously, are not useful as far as the method of eigenfunction expansion is concerned. Class X gives a periodic potential, which is also not of concern in this paper, as we are interested in normalizable wave functions so that the distribution density of the FP equation is also normalizable.

So we are left with only classes I, IV, VI and VII in the list of [3] which give QES FP systems having  $N + 1$  eigenstates with arbitrary  $N$ . To obtain these four QES FP systems, one needs only to identify  $W_0$  with the gauge function  $g$  of each class listed in [3]. This gives a QES FP system with only one solvable eigenstate. For QES FP systems in the same class but with higher values of  $N$ , we construct  $W_N$  from  $W_0$  and the roots of the Bethe ansatz equations  $\{z_k\}$  (with the function  $z = z(x)$  also from [3]) according to the procedure in [8].

Let us illustrate this by class VII QES FP system. It turns out that this case is just the one-particle case of the so-called rational  $BC$  type Inozemtsev model [7].

In this case  $z(x) = x^2$ , and the prepotential (9) is given by

$$W_N(x) = W_0 + \bar{W}, \quad (17)$$

$$W_0 = \frac{a}{4}x^4 + \frac{b}{2}x^2 - \gamma \ln x + C_N, \quad x > 0, \quad a > 0, \quad \gamma > 0, \quad (18)$$

$$\bar{W} = - \sum_{k=1}^N \ln |x^2 - z_k|. \quad (19)$$

Here  $C_N$  is a constant term necessary for the normalization of the ground state wave function  $\phi_N^{(0)}(x)$ . By plugging in all  $N+1$  possible solutions  $\{z_k\}$  into  $V_N$  in (10), we find that  $V_N$  is independent of  $\{z_k\}$ . The Bethe ansatz equations removing the simple poles at  $\{z_k\}$  read

$$2az_k^2 + 2bz_k - (2\gamma + 1) - 4 \sum_{l \neq k} \frac{z_k}{z_k - z_l} = 0, \quad k = 1, \dots, N, \quad (20)$$

and  $\lambda_N$  in terms of the roots  $z_k$ 's is

$$\lambda_N = 2(2\gamma + 1) \sum_{k=1}^N \frac{1}{z_k}, \quad (21)$$

or equivalently, using the Bethe ansatz equation,

$$\lambda_N = 4bN + 4a \sum_{k=1}^N z_k. \quad (22)$$

Among all the solutions, we choose those  $\{z_k\}$  which are all negative

$$z_k < 0, \quad k = 1, \dots, N. \quad (23)$$

This corresponds to the ground state, since the eigenvalue  $\lambda_N^{(0)}$  is the lowest, and the wave function

$$\phi_N^{(0)} = e^{-(\frac{a}{4}x^4 + \frac{b}{2}x^2)} x^\gamma \prod_{k=1}^N (x^2 - z_k) \quad (24)$$

has no node. The fact that the choice (23) is possible can be understood as follows. Recall that the number of QES states is  $N+1$  for each  $N$ . Since the factor  $\prod_{k=1}^N (x^2 - z_k)$  in the wave function (24) is even in  $x$ , there can be at most  $N$  zeros in the half line  $x > 0$ . According to a well known theorem in quantum mechanics, no two eigenstates can have the same number of nodes. This implies that the number of nodes of the  $N+1$  QES states here must be  $0, 1, 2, \dots, N$ . Hence the ground state is in the QES sector, and the choice (23) is guaranteed. A similar argument can be applied to the other cases.

The corresponding QES FP equation is then defined by the drift potential

$$\Phi_N = \frac{a}{2}x^4 + bx^2 - 2\gamma \ln x - 2 \sum_{k=1}^N \ln |x^2 - z_k| + \text{constant}. \quad (25)$$

The constant is chosen so that the stationary distribution of this FP equation  $\phi_N^{(0)2} = \exp(-\Phi_N)$  is normalized. We have thus obtained a QES FP equation corresponding to the class VII  $sl(2)$ -based QES Schrödinger equation in [3].

QES FP equations of the other three classes can be treated accordingly. The main defining information is given in Table 1.

### C. $sl(2)$ -based QES case not listed in Turbiner's classification

We now present a QES case which is also based on the  $sl(2)$  algebra, but not listed in [3]. This is related to the trigonometric  $BC$  type Inozemtsev model with one degree of freedom [7].

TABLE I: Three other QES FP systems associated with the  $sl(2)$ -based QES Schrödinger equations in [3]. The function  $z(x)$ , prepotential  $W_0(x)$  ( $-\infty < x < \infty$ ), Bethe ansatz equations, and the eigenvalues  $\lambda_N$  are listed. The drift potential defining the QES FP equation with  $N + 1$  solvable states is  $\Phi_N = 2W_N$ , where the prepotential  $W_N$  is constructed from (9) with  $W_0$  and the roots of the Bethe ansatz equations. All parameters are real.

Class	$z(x)$	$W_0$	Bethe ansatz equations ( $k = 1, 2, \dots, N$ )	$\lambda_N$
I	$e^{-\alpha x}$	$\frac{a}{\alpha}e^{-\alpha x} + bx + \frac{c}{\alpha}e^{\alpha x}$ ( $\alpha > 0, a, c \geq 0, \forall b$ )	$az_k^2 - (b + \frac{a}{2})z_k - c$ $-\alpha \sum_{l \neq k} \frac{z_k^2}{z_k - z_l} = 0$	$2\alpha c \sum_{k=1}^N \frac{1}{z_k}$
IV	$\cosh^{-2} \alpha x$	$\frac{c}{4\alpha} \cosh 2\alpha x + \frac{a}{\alpha} \ln \cosh \alpha x$ ( $\alpha > 0, c \geq 0, \forall a$ )	$(a + \frac{3}{2}\alpha)z_k^2 + (c - a - \alpha)z_k - c$ $-2\alpha \sum_{l \neq k} \frac{z_k^2(1-z_k)}{z_k - z_l} = 0$	$2ac - a^2 - \alpha c$ $+4\alpha c \sum_{k=1}^N \frac{1}{z_k}$
VI	$x^2$	$\frac{a}{4}x^4 + \frac{b}{2}x^2$ ( $a > 0, \forall b$ or $a \geq 0, b > 0$ )	$az_k^2 + bz_k - \frac{1}{2}$ $-2\alpha \sum_{l \neq k} \frac{z_k}{z_k - z_l} = 0$	$2 \sum_{k=1}^N \frac{1}{z_k}$

The prepotential in this case reads

$$W_N = \frac{a}{2} \cos 2x - \frac{b}{2} \ln |\cot x| - c \log |\sin x| - \sum_{k=1}^N \ln |\sin^2 x - z_k| + C_N, \quad 0 < x < \frac{\pi}{2}. \quad (26)$$

All the parameters  $a, b$  and  $c$  are real and satisfy  $a > 0, c > \frac{b}{2} > 0$ . Again,  $C_N$  is a constant term necessary for the normalization of the ground state wave function  $\phi_N^{(0)}(x)$ . The first three terms in (26) are just  $W_0$  which gives only the ground state solution. The parameters  $\{z_k\}$  are determined by the following Bethe ansatz equations:

$$(4az_k + 2c)(1 - z_k) - b + 1 - 2z_k + 4z_k(1 - z_k) \sum_{l \neq k} \frac{1}{z_k - z_l} = 0, \quad k = 1, \dots, N. \quad (27)$$

With these solutions, the potential  $V_N$  and eigenvalue  $\lambda_N$  in the Riccati equation (10) are

$$V_N(x, \{z_k\}) = \left( \frac{dW_0}{dx} \right)^2 - \frac{d^2W_0}{dx^2} - 8aN \sin^2 x, \quad (28)$$

and

$$\lambda_N(\{z_k\}) = 8a \sum_{k=1}^N z_k + 4N(N + c). \quad (29)$$

Again we choose those  $\{z_k\}$  which are all negative

$$z_k < 0, \quad k = 1, \dots, N.$$

This corresponds to the ground state and its wave function has no node:

$$e^{-W_N^{(0)}(x)} \propto e^{-\frac{a}{2} \cos 2x} (\cot x)^{\frac{b}{2}} (\sin x)^c \prod_{k=1}^N (\sin^2 x - z_k). \quad (30)$$

The corresponding QES FP equation is then defined by the drift potential  $\Phi_N = 2W_N^{(0)}$ .

The prepotential  $W_N$  in (26), and hence the potential  $V_N$  in (28) are not listed in the  $sl(2)$ -based classification scheme in [3]. Nevertheless, it turns out that this system is also related to  $sl(2)$  Lie algebra [13]. This can be seen as follows. Using  $\phi_0 = \exp(-W_0)$  and  $V_N$ , we obtain the transformed Hamiltonian  $h_N = \phi_0^{-1} \mathcal{H}_N \phi_0$  as

$$\begin{aligned} h_N &= -\frac{d^2}{dx^2} - 2\frac{dW_0}{dx} \frac{d}{dx} - 8aN \sin^2 x, \\ &= -\frac{d^2}{dx^2} - 2 \left( a \sin 2x - \frac{b}{\sin 2x} + c \cot x \right) \frac{d}{dx} - 8aN \sin^2 x. \end{aligned} \quad (31)$$

In terms of the new variable  $z(x) = \sin^2 x$ ,  $h_N$  becomes

$$h_N = -4z(1-z) \frac{d^2}{dz^2} + 2 \left[ 4az^2 + 2(c-2a+1)z + b-2c-1 \right] \frac{d}{dz} - 8aNz. \quad (32)$$

Now it is easy to check that eq. (32) can in fact be rewritten in the form (15) as

$$h_N = 4J^+ J^- - 4J^0 J^- + 8aJ^+ + 4(c-2a+1+N)J^0 + 2(b-2c-1-N)J^- + 2N(c-2a+1+N), \quad (33)$$

where the generators of the Lie algebra  $sl(2)$  are given by

$$\begin{aligned} J^+ &= z^2 \frac{d}{dz} - Nz, \\ J^0 &= z \frac{d}{dz} - \frac{N}{2}, \\ J^- &= \frac{d}{dz}. \end{aligned} \quad N = 0, 1, 2, \dots \quad (34)$$

This proves that the trigonometric  $BC$  type Inozemtsev model is indeed a QES system based on  $sl(2)$ .

## V. SUMMARY

In summary, we have discussed exact and quasi-exact solvability of the FP equation based on the corresponding solvability of its associated Schrödinger equation. We give a unified treatment of these two types of solvability from the viewpoint of prepotential together with Bethe ansatz equations. Examples of QES FP equations related to the  $sl(2)$ -based systems listed in [3] are classified, and one example which is also based on the  $sl(2)$  algebra but not listed in [3] is also presented. We note here that, as shown in [7], the concepts of quasi-exact solvability and the so-called higher derivative, or nonlinear, or  $\mathcal{N}$ -fold supersymmetry [14] are equivalent. Hence, all the QES FP equations discussed here also possess such kind of extended supersymmetry.

As mentioned before, we only discuss those QES FP systems which can admit a large number of QES states so that they are useful for consideration when approximate calculations are required. Hence, we have not considered the so-called second type QES systems defined in [3], as they admit only a single solvable state and are therefore of less interest in practical calculations. Nevertheless, these QES quantum systems still give the corresponding QES FP equations. We hope that in the future more non- $sl(2)$  QES FP systems can be found.

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